

Vanishing of the first continuous L^2 -cohomology for II_1 factors

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Abstract

We prove that the continuous version of the Connes-Shlyakhtenko first L^2 -cohomology for II_1 factors, as proposed by A. Thom in [Th06], always vanishes.

In [CS03], A. Connes and D. Shlyakhtenko developed an L^2 -cohomology theory for finite von Neumann algebras M , and more generally for weakly dense $*$ -subalgebras $A \subset M$ of such von Neumann algebras. Then in [Th06], A. Thom provided an alternative, Hochschild-type characterization of the first such L^2 -cohomology of M as the quotient of the space of derivations $\delta : M \rightarrow \text{Aff}(M \overline{\otimes} M^{\text{op}})$ by the space of inner derivations, where $\text{Aff}(M \overline{\otimes} M^{\text{op}})$ denotes the $*$ -algebra of operators affiliated with $M \overline{\otimes} M^{\text{op}}$. Thom also proposed in [Th06] a continuous version of the first L^2 -cohomology, by considering the (smaller) space of derivations δ that are continuous from M with the operator norm to $\text{Aff}(M \overline{\otimes} M^{\text{op}})$ with the topology of convergence in measure. He noted that in many cases (e.g., when M has a Cartan subalgebra, or when M is not prime), this cohomology vanishes, i.e. any continuous derivation of M into $\text{Aff}(M \overline{\otimes} M^{\text{op}})$ is inner.

Following up on this work, V. Alekseev and D. Kyed have shown in [AK11] that the first continuous L^2 -cohomology also vanishes when M has property (T), when M is finitely generated with nontrivial fundamental group, or when M has property Gamma. Recently, V. Alekseev proved in [Al13] that this is also the case for the free group factors $L(\mathbb{F}_n)$.

In this article, we prove that in fact the first continuous L^2 -cohomology vanishes for all finite von Neumann algebras. The starting point of our proof is a key calculation in the proof of [Al13, Proposition 3.1], which provides a concrete sequence of elements y_n in the II_1 factor $M = L(\mathbb{F}_3)$ of the free group \mathbb{F}_3 with generators a, b, c , that tends to 0 in operator norm, but has the property that if a derivation $\delta : M \rightarrow \text{Aff}(M \overline{\otimes} M^{\text{op}})$ satisfies $\delta(u_a) = u_a \otimes 1$ and $\delta(u_b) = \delta(u_c) = 0$, then $\delta(y_n)$ does not tend to 0 in measure. More precisely, the y_n 's in [Al13] are scalar multiples of words w_n in a, b, c with the property that $\delta(w_n)$ is a larger and larger sum of free independent Haar unitaries. In the case of an arbitrary II_1 factor M , we fix a hyperfinite II_1 factor $R \subset M$ with trivial relative commutant, and then use [Po92] to “simulate” (in distribution) $L(\mathbb{F}_3)$ inside M , with a any fixed unitary in M and b_m, c_m Haar unitaries in R such that a, b_m, c_m are asymptotically free. If now δ is a continuous derivation on M , then by subtracting an inner derivation, we may assume δ vanishes on R , thus on b_m, c_m . If $\delta(a) \neq 0$, and if we formally define y_n 's via the same formula as Alekseev's, with a, b_m, c_m in lieu of a, b, c , then a careful estimation of norms of y_n and $\delta(y_n)$, which uses results in [HL99], shows that one still has $\|y_n\| \rightarrow 0$, while $\delta(y_n) \not\rightarrow 0$ in measure.

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Let M be a finite von Neumann algebra. We denote by M^{op} the opposite von Neumann algebra and by $\text{Aff}(M \overline{\otimes} M^{\text{op}})$ the $*$ -algebra of operators affiliated with $M \overline{\otimes} M^{\text{op}}$. A *derivation* $\delta : M \rightarrow \text{Aff}(M \overline{\otimes} M^{\text{op}})$ is a linear map satisfying

$$\delta(ab) = (a \otimes 1)\delta(b) + (1 \otimes b^{\text{op}})\delta(a) \quad \text{for all } a, b \in M.$$

For every $\xi \in \text{Aff}(M \overline{\otimes} M^{\text{op}})$, denote by $\partial\xi$ the *inner* derivation defined as

$$(\partial\xi)(a) = (a \otimes 1 - 1 \otimes a^{\text{op}})\xi \quad \text{for all } a \in M.$$

We equip $\text{Aff}(M \overline{\otimes} M^{\text{op}})$ with the *measure topology*, i.e. the unique vector space topology with basic neighborhoods of 0 given by

$$B(\tau_1, \varepsilon) = \{\xi \in \text{Aff}(M \overline{\otimes} M^{\text{op}}) \mid \exists \text{ projection } p \in M \overline{\otimes} M^{\text{op}} \text{ with } \tau_1(p) > 1 - \varepsilon, \|\xi p\| < \varepsilon\}$$

for all normal tracial states $\tau_1 : M \rightarrow \mathbb{C}$ and all $\varepsilon > 0$. If $\tau : M \rightarrow \mathbb{C}$ is a normal *faithful* tracial state, then $\{B(\tau, \varepsilon) \mid \varepsilon > 0\}$ is a family of basic neighborhoods of 0 and there is no need to vary the trace.

Theorem 1. *Let M be a finite von Neumann algebra. Every derivation $\delta : M \rightarrow \text{Aff}(M \overline{\otimes} M^{\text{op}})$ that is continuous from the norm topology on M to the measure topology on $\text{Aff}(M \overline{\otimes} M^{\text{op}})$, is inner.*

The following lemma is quite standard, but we include a detailed proof for completeness.

Lemma 2. *It suffices to prove Theorem 1 for II_1 factors M with separable predual.*

Proof. We prove the lemma in different steps.

Step 1. It suffices to prove Theorem 1 for diffuse, countably decomposable M . Take a set $I = I_1 \sqcup I_2$ and an orthogonal family of projections $p_i \in \mathcal{Z}(M)$ with $\sum_{i \in I} p_i = 1$ and such that for all $i \in I_1$, we have that Mp_i is countably decomposable and diffuse, and such that for all $i \in I_2$, we have that Mp_i is a matrix algebra. Since the projections p_i are orthogonal, the element

$$\xi = \sum_{i \in I} (p_i \otimes 1)\delta(p_i)$$

is well defined in $\text{Aff}(M \overline{\otimes} M^{\text{op}})$. The following direct computation shows that

$$\delta(p_k) = (\partial\xi)(p_k) \quad \text{for all } k \in I. \tag{1}$$

Indeed,

$$(\partial\xi)(p_k) = (p_k \otimes 1)\delta(p_k) - \sum_{i \in I} (p_i \otimes p_k^{\text{op}})\delta(p_i). \tag{2}$$

But, for all i and k , we have

$$(1 \otimes p_k^{\text{op}})\delta(p_i) = \delta(p_i p_k) - (p_i \otimes 1)\delta(p_k).$$

Multiplying with $p_i \otimes 1$ and summing over i , we find that

$$\sum_{i \in I} (p_i \otimes p_k^{\text{op}})\delta(p_i) = (p_k \otimes 1)\delta(p_k) - \delta(p_k).$$

In combination with (2), we find that (1) holds.

Replacing δ by $\delta - \partial\xi$, we may assume that $\delta(p_i) = 0$ for all $i \in I$. It follows that $\delta(Mp_i) \subset \text{Aff}(Mp_i \overline{\otimes} (Mp_i)^{\text{op}})$ for every $i \in I$. We denote by δ_i the restriction of δ to Mp_i . By the assumption of step 1, δ_i is inner when $i \in I_1$. So for $i \in I_1$, we have that $\delta_i = \partial\xi_i$ for some $\xi_i \in \text{Aff}(Mp_i \overline{\otimes} (Mp_i)^{\text{op}})$. When $i \in I_2$, we have that Mp_i is a matrix algebra and we can take a complete system of matrix units e_{jk}^i for Mp_i . We then get that $\delta_i = \partial\xi_i$ for

$$\xi_i = \sum_k (e_{k1}^i \otimes 1) \delta_i(e_{1k}^i).$$

The vector $\xi = \sum_{i \in I} \xi_i$ is well defined in $\text{Aff}(M \overline{\otimes} M^{\text{op}})$ and $\delta = \partial\xi$.

Step 2. It suffices to prove Theorem 1 when M is diffuse and has separable predual. Using step 1, we may already assume that M is diffuse and admits a faithful normal tracial state τ that we keep fixed. We start by proving the following three statements, using that M admits the faithful trace τ .

If $\xi \in \text{Aff}(M \overline{\otimes} M^{\text{op}})$, there exists a von Neumann subalgebra $N \subset M$ with separable predual such that $\xi \in \text{Aff}(N \overline{\otimes} N^{\text{op}})$. Take an increasing sequence of projections $p_n \in M \overline{\otimes} M^{\text{op}}$ such that $\tau(1 - p_n) \rightarrow 0$ and $\xi p_n \in M \overline{\otimes} M^{\text{op}}$ for all n . In particular, $\xi p_n \in L^2(M \overline{\otimes} M^{\text{op}}) = L^2(M) \otimes L^2(M^{\text{op}})$ and we can take separable Hilbert subspaces $H_n \subset L^2(M)$, $K_n \subset L^2(M^{\text{op}})$ such that $\xi p_n \in H_n \otimes K_n$. We then find countable subsets $V_n \subset M$ such that for every n , the vector ξp_n belongs to the $\|\cdot\|_2$ -closed linear span of $\{a \otimes b^{\text{op}} \mid a, b \in V_n\}$. Defining N as the von Neumann subalgebra of M generated by all the sets V_n , our statement is proven.

Let $N_1 \subset M$ be a von Neumann subalgebra with separable predual. Then there exists a von Neumann subalgebra $N_2 \subset M$ with separable predual such that $N_1 \subset N_2$ and $\delta(N_1) \subset \text{Aff}(N_2 \overline{\otimes} N_2^{\text{op}})$. Take a separable and weakly dense C^* -subalgebra $B_1 \subset N_1$. By the previous paragraph and because δ is norm-measure continuous, we can take $N_2 \subset M$ with separable predual such that $\delta(B_1) \subset \text{Aff}(N_2 \overline{\otimes} N_2^{\text{op}})$. Replacing N_2 by the von Neumann algebra generated by N_1 and N_2 , we may assume that $N_1 \subset N_2$. Since $\delta(B_1) \subset \text{Aff}(N_2 \overline{\otimes} N_2^{\text{op}})$, it follows from [Th06, Lemma 4.2 and Theorem 4.3] that $\delta(N_1) \subset \text{Aff}(N_2 \overline{\otimes} N_2^{\text{op}})$.

Let $N_1 \subset M$ be a von Neumann subalgebra with separable predual. Then there exists a von Neumann subalgebra $N \subset M$ with separable predual such that $N_1 \subset N$ and $\delta(N) \subset \text{Aff}(N \overline{\otimes} N^{\text{op}})$. Using the previous paragraph, we inductively find an increasing sequence of von Neumann subalgebras $N_1 \subset N_2 \subset \dots$ with separable predual such that $\delta(N_k) \subset \text{Aff}(N_{k+1} \overline{\otimes} N_{k+1}^{\text{op}})$ for all k . We define N as the von Neumann algebra generated by all the N_k . By construction, N has separable predual and $\delta(N_k) \subset \text{Aff}(N \overline{\otimes} N^{\text{op}})$ for all k . Again using [Th06, Lemma 4.2 and Theorem 4.3], it follows that $\delta(N) \subset \text{Aff}(N \overline{\otimes} N^{\text{op}})$.

We can now conclude the proof of step 2. Since M is diffuse, we can fix a diffuse abelian von Neumann subalgebra $A \subset M$ with separable predual. By [Th06, Theorem 6.4], we can replace δ by $\delta - \partial\xi$ for some $\xi \in \text{Aff}(M \overline{\otimes} M^{\text{op}})$ and assume that $\delta(a) = 0$ for all $a \in A$. We prove that $\delta(x) = 0$ for all $x \in M$. Fix an arbitrary $x \in M$. Define N_1 as the von Neumann algebra generated by A and x . Note that N_1 has a separable predual. By the previous paragraph, we can take a von Neumann subalgebra $N \subset M$ with separable predual such that $N_1 \subset N$ and $\delta(N) \subset \text{Aff}(N \overline{\otimes} N^{\text{op}})$. By the initial assumption of step 2, the restriction of δ to N is inner. So we can take a $\xi \in \text{Aff}(N \overline{\otimes} N^{\text{op}})$ such that $\delta(y) = (\partial\xi)(y)$ for all $y \in N$. Since $A \subset N$ and $\delta(a) = 0$ for all $a \in A$, it follows that $(a \otimes 1)\xi = (1 \otimes a^{\text{op}})\xi$ for all $a \in A$. Since A is diffuse, this implies that $\xi = 0$. Since $x \in N$, it then follows that $\delta(x) = 0$. This concludes the proof of step 2.

Step 3. Proof of the lemma : it suffices to prove Theorem 1 when M is a II_1 factor with separable predual. Using step 2, we may already assume that M is diffuse and has separable predual.

Let $p_0 \in \mathcal{Z}(M)$ be the maximal projection such that $\mathcal{Z}(M)p_0$ is diffuse (possibly, $p_0 = 0$). Let p_1, p_2, \dots be the minimal projections in $\mathcal{Z}(M)(1 - p_0)$. Note that $\sum_{n=0}^{\infty} p_n = 1$. As in the proof of step 1, we may assume that $\delta(Mp_n) \subset \text{Aff}(Mp_n \overline{\otimes} (Mp_n)^{\text{op}})$ for all n . Denote by δ_n the restriction of δ to Mp_n . Since Mp_0 has a diffuse center, it follows from [Th06, Theorem 6.4] that δ_0 is inner. For all $n \geq 1$, we have that Mp_n is a II_1 factor with separable predual. So by assumption, all δ_n , $n \geq 1$, are inner. But then also δ is inner. \square

Proof of Theorem 1. Using Lemma 2, it suffices to take a II_1 factor M with separable predual and a derivation $\delta : M \rightarrow \text{Aff}(M \overline{\otimes} M^{\text{op}})$ that is continuous from the norm topology on M to the measure topology on $\text{Aff}(M \overline{\otimes} M^{\text{op}})$. Denote by τ the unique tracial state on M . By [Po81, Corollary 4.1], we can fix a copy of the hyperfinite II_1 factor $R \subset M$ such that $R' \cap M = \mathbb{C}1$. By [Th06, Theorem 6.4], we can replace δ by $\delta - \partial\xi$ and assume that $\delta(x) = 0$ for all $x \in R$. We prove that $\delta = 0$. Fix a unitary $u \in \mathcal{U}(M)$ with $\tau(u) = 0$. It suffices to prove that $\delta(u) = 0$.

Fix a free ultrafilter ω on \mathbb{N} and consider the ultrapower M^ω . By [Po92, Corollary on p. 187], choose a unitary $v \in R^\omega$ such that the subalgebras $v^k M v^{-k} \subset M^\omega$, $k \in \mathbb{Z}$, are free. Fix a Haar unitary $a \in \mathcal{U}(R)$, i.e. a unitary satisfying $\tau(a^m) = 0$ for all $m \in \mathbb{Z} \setminus \{0\}$. Define, for $k \geq 1$, $a_k = v^k a v^{-k}$. It follows that $a_k \in \mathcal{U}(R^\omega)$ are $*$ -free Haar unitaries that are moreover free w.r.t. M . Write $a_k = (a_{k,n})$ with $a_{k,n} \in \mathcal{U}(R)$.

Similar to the definition of x_n in the proof of [Al13, Proposition 3.1], we consider for all large n and m , the unitary

$$w_{m,n} = a_{1,n} u a_{2,n} u \cdots a_{m,n} u$$

and prove that either $\delta(u) = 0$ or $\delta(w_{m,n})$ is “very large almost everywhere”, contradicting the continuity of δ .

Since $\delta(x) = 0$ for all $x \in R$, we get that

$$\delta(w_{m,n}) = \left(\sum_{k=1}^m a_{1,n} u a_{2,n} \cdots a_{k-1,n} u a_{k,n} \otimes (a_{k+1,n} u \cdots a_{m,n} u)^{\text{op}} \right) \delta(u), \quad (3)$$

where, by convention, the first term in the sum is $a_{1,n} \otimes (a_{2,n} u \cdots a_{m,n} u)^{\text{op}}$ and the last term is $a_{1,n} u \cdots a_{m-1,n} u a_{m,n} \otimes 1$.

Consider, in the ultrapower $(M \overline{\otimes} M^{\text{op}})^\omega$, the element

$$T_m = \sum_{k=1}^m a_1 u a_2 u \cdots a_{k-1} u a_k \otimes (a_{k+1} u \cdots a_m u)^{\text{op}}.$$

We claim that T_m is the sum of m $*$ -free Haar unitaries. To prove this, it suffices to show that the first tensor factors $a_1, a_1 u a_2, a_1 u a_2 u a_3, \dots$ form a $*$ -free family of Haar unitaries. Since a_1, a_2, a_3, \dots is a $*$ -free family of Haar unitaries that are $*$ -free w.r.t. u , also the Haar unitaries $a_1, u a_2, u a_3, u a_4, \dots$ are $*$ -free. But then the conclusion follows by taking the product of the first k unitaries in this last sequence, again producing a $*$ -free family of Haar unitaries.

Since T_m is the sum of m $*$ -free Haar unitaries, we get from [HL99, Example 5.5] an explicit formula for the spectral distribution of $|T_m|$. It follows that $|T_m|$ has the same distribution as $2\sqrt{m-1} S_m$, where S_m is a sequence of random variables satisfying $0 \leq S_m \leq 1$ and converging in distribution to the normalized quarter circle law. Therefore, the spectral projections

$$q_m = \mathbf{1}_{[\sqrt{m}, +\infty)}(T_m^* T_m) = \mathbf{1}_{[m^{1/4}, +\infty)}(|T_m|) = \mathbf{1}_{[m^{1/4}(4(m-1))^{-1/2}, +\infty)}(S_m)$$

satisfy $\lim_m \tau(q_m) = 1$. Write $q_m = (q_{m,n})$ where $q_{m,n}$ are projections in $M \overline{\otimes} M^{\text{op}}$.

Fix an arbitrary $\varepsilon > 0$. Since δ is continuous, fix $\rho > 0$ such that $\delta(z) \in B(\tau, \varepsilon/2)$ whenever $z \in M$ and $\|z\| < \rho$. Take m large enough such that $m^{-1/4} < \rho$ and $\tau(q_m) > 1 - \varepsilon$. For every n , the element $m^{-1/4}w_{m,n}$ has norm less than ρ . Therefore, $\delta(m^{-1/4}w_{m,n})$ belongs to $B(\tau, \varepsilon/2)$ and we find a projection $p_n \in M \overline{\otimes} M^{\text{op}}$ with

$$\tau(p_n) > 1 - \varepsilon/2 \quad \text{and} \quad \|\delta(m^{-1/4}w_{m,n}) p_n\| < \varepsilon/2.$$

We also fix a projection $e_0 \in M \overline{\otimes} M^{\text{op}}$ with $\tau(e_0) > 1 - \varepsilon/2$ and such that $\delta(u)e_0 \in M \overline{\otimes} M^{\text{op}}$. We write $e_n = e_0 \wedge p_n$ and view $e = (e_n)$ as a projection in $(M \overline{\otimes} M^{\text{op}})^\omega$. By (3), we have in $(M \overline{\otimes} M^{\text{op}})^\omega$ the equality $(\delta(w_{m,n})e_0)_n = T_m(\delta(u)e_0)$, and therefore also that $(\delta(w_{m,n})e_n)_n = T_m(\delta(u)e_0)e$. We then find that

$$\begin{aligned} \varepsilon^2 &> \lim_{n \rightarrow \omega} \|\delta(m^{-1/4}w_{m,n}) e_n\|^2 \\ &\geq \lim_{n \rightarrow \omega} \|\delta(m^{-1/4}w_{m,n}) e_n\|_2^2 \\ &= \tau(e (\delta(u)e_0)^* m^{-1/2} T_m^* T_m (\delta(u)e_0) e) \\ &\geq \tau(e (\delta(u)e_0)^* q_m (\delta(u)e_0) e). \end{aligned}$$

Since $\tau(q_m) > 1 - \varepsilon$, we can fix n such that

$$\|q_{m,n} \delta(u) e_n\|_2 < \varepsilon \quad \text{and} \quad \tau(q_{m,n}) > 1 - \varepsilon.$$

Since $\tau(e_n) > 1 - \varepsilon$, we have proven that for every $\varepsilon > 0$, there exist projections $p, q \in M \overline{\otimes} M^{\text{op}}$ such that $\tau(p) > 1 - \varepsilon$, $\tau(q) > 1 - \varepsilon$ and $\|q\delta(u)p\|_2 < \varepsilon$. This means that $\delta(u) = 0$. \square

The proof of Theorem 1 gives no indication as to whether or not the Connes-Shlyakhtenko first L^2 -cohomology vanishes as well. Note however that in order for a first L^2 -cohomology theory to “work well” for II_1 factors M , the corresponding derivations should be uniquely determined by their values on a set of elements generating M as a von Neumann algebra. In order for this to be the case, the derivations should normally satisfy some continuity property, even if that continuity is “very weak”. However, by combining Theorem 1 with the closed graph theorem, it follows that any derivation from M into $\text{Aff}(M \overline{\otimes} M^{\text{op}})$ that satisfies some “reasonable” weak continuity property, must in fact be inner (see also Remark 4 hereafter):

Corollary 3. *Let M be a finite von Neumann algebra and write $\mathcal{E} = \text{Aff}(M \overline{\otimes} M^{\text{op}})$. Assume that $\delta : M \rightarrow \mathcal{E}$ is a derivation. If δ has a closed graph for the norm topology on M and the measure topology on \mathcal{E} , then δ is inner.*

This is in particular the case if δ is norm- \mathcal{T} -continuous w.r.t. any vector space topology \mathcal{T} on \mathcal{E} satisfying the following two properties:

- *the inclusion $M \overline{\otimes} M^{\text{op}} \rightarrow \mathcal{E}$ is norm- \mathcal{T} -continuous;*
- *for every fixed $a \in M \overline{\otimes} M^{\text{op}}$, the map $\mathcal{E} \rightarrow \mathcal{E} : \xi \mapsto \xi a$ is \mathcal{T} - \mathcal{T} -continuous.*

Proof. Take an orthogonal family of projections $p_i \in \mathcal{Z}(M)$ such that $\sum_{i \in I} p_i = 1$ and every Mp_i is countably decomposable. As in step 1 of Lemma 2, we may assume that $\delta(Mp_i) \subset \text{Aff}(Mp_i \overline{\otimes} (Mp_i)^{\text{op}})$. If δ has closed graph, the restrictions δ_i of δ to Mp_i still have closed graph. If all these restrictions δ_i are inner, also δ is inner. So to prove the first part of the corollary, we may assume that M admits a faithful normal tracial state τ that we keep fixed. But then, the formula

$$d(\xi, \eta) = \inf\{\varepsilon > 0 \mid \exists \text{ projection } p \in M \overline{\otimes} M^{\text{op}} : \tau(1 - p) < \varepsilon \text{ and } \|(\xi - \eta)p\| < \varepsilon\}$$

defines a translation invariant, complete metric on $\text{Aff}(M \overline{\otimes} M^{\text{op}})$ that induces the measure topology. So by [Ru91, 2.15], the closed graph theorem is valid and we find that δ is continuous, and hence inner by Theorem 1.

Assume now in general that δ is norm- \mathcal{T} -continuous. To prove that δ has closed graph, assume that $\|x_n\| \rightarrow 0$ and $\delta(x_n) \rightarrow \xi$ in the measure topology. We have to show that $\xi = 0$. Fix a normal tracial state τ on M . Choose projections $p_n \in M \overline{\otimes} M^{\text{op}}$ such that $\tau(p_n) > 1 - 2^{-n}$ and $\|(\delta(x_n) - \xi)p_n\| < 1/n$. Define the projections $q_k = \bigwedge_{n \geq k} p_n$ and note that q_k is increasing and satisfies $\tau(q_k) \rightarrow 1$. For every fixed k , we get that $(\delta(x_n) - \xi)q_k$ converges to 0 in norm, as $n \rightarrow \infty$. By the first assumption on \mathcal{T} , this convergence also holds in \mathcal{T} . But also $\delta(x_n) \rightarrow 0$ in \mathcal{T} so that, by our second assumption on \mathcal{T} , the sequence $\delta(x_n)q_k$ converges to 0 in \mathcal{T} as $n \rightarrow \infty$. We conclude that $\xi q_k = 0$ for all k . This implies that $\xi z_\tau = 0$ where $z_\tau \in \mathcal{Z}(M)$ is the support projection of τ . Since τ was arbitrary, it follows that $\xi = 0$. \square

Remark 4. We should point out that we have no concrete examples of vector topologies on $\text{Aff}(M \overline{\otimes} M^{\text{op}})$ satisfying the conditions in the second part of Corollary 3 and that are strictly weaker than the measure topology (in fact, it is not even clear whether such a topology exists!). Let us also point out that there are other weak continuity properties of δ implying that δ has closed graph, thus following inner by the first part of Corollary 3. For instance, by using a similar argument as above, one can easily prove that this is the case when δ satisfies the following weak continuity property: whenever x_n is a sequence in M such that $\|x_n\| \rightarrow 0$, there exists a sequence of projections $p_n \in M \overline{\otimes} M^{\text{op}}$ such that $p_n \rightarrow 1$ strongly, $\delta(x_n)p_n \in M \overline{\otimes} M^{\text{op}}$ and $\delta(x_n)p_n \rightarrow 0$ σ -weakly.

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